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2003 J. Phys. A: Math. Gen. 36 3067

(http://iopscience.iop.org/0305-4470/36/12/313)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 3067–3083 PII: S0305-4470(03)36991-4

Partition functions for matrix models and isomonodromic tau functions

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Received 15 May 2002
Published 12 March 2003
Online at stacks.iop.org/JPhysA/36/3067

Abstract

For one-matrix models with polynomial potentials, the explicit relationship between the partition function and the isomonodromic tau function for the 2×2 polynomial differential systems satisfied by the associated orthogonal polynomials is derived.

PACS numbers: 02.10.Yn, 02.30.Ik

1. Introduction

It is well known [15, 27, 34] that the partition functions for matrix models, for both finite $n \times n$ random matrices and suitable $n \to \infty$ limits, provide τ functions for the KP hierarchy. It is also known that such partition functions satisfy certain constraints amounting to the invariance of the corresponding elements in the associated infinite Grassmannian under a set of positive Virasoro generators. This generally implies that there exist differential operators in the spectral parameter variable which annihilate the associated Baker–Akhiezer functions. The compatibility of the differential equations in the spectral parameter and deformation parameters representing KP-flows implies that the monodromy of the operator in the spectral parameter is invariant under such deformations. In the large n limit this was considered in [26], while for finite n it was studied in [21]. Associated with the analyticity properties of the Baker–Akhiezer functions is an isomonodromic tau function [22], defined similarly as the KP tau function.

In practice, the exact relationship between such 'isomonodromic' tau functions and the KP tau function is not very clearly understood. There are special cases, however (see, e.g., [1–3, 14, 16, 17, 20–22, 26]) where these two quantities can either be shown to be identical, or at least explicitly related. In the computation of spacing distributions for random matrices, a class of isomonodromic tau functions occurs, given by Fredholm determinants of 'integrable' integral operators [9, 17, 18, 28, 32, 33] supported on a union of intervals, with the deformation parameters taken as the endpoints of the intervals rather than as KP flow parameters. The precise relation between such deformations and the KP flows is not yet clearly understood. (See, however, [1, 2, 16, 34] for some indications of the links between the two.)

It is our purpose here to analyse in detail, for a generalization of the simplest version of the Hermitian one-matrix model, with measures that are exponentials of polynomial trace invariants, exactly what this relationship is. The main idea is to consider, for finite n (the size of the random matrix), the isomonodromic deformation systems satisfied by the associated sequence of orthogonal polynomials, and to compute an explicit formula (theorem 2.4) relating the associated isomonodromic tau function to the partition function of the generalized matrix model. In the process, we recall (and generalize) some of the standard results concerning the relation of isomonodromic deformations to random matrices and the occurrence of Stokes phenomena in solutions of the associated systems of differential equations with polynomial coefficients.

2. Orthogonal polynomials for semiclassical functionals

We consider the orthogonal polynomials for a moment functional

$$\mathcal{L}_{\varkappa\Gamma} : \mathbb{C}[x] \mapsto \mathbb{C} \tag{2.1a}$$

$$\mathcal{L}_{\varkappa\Gamma}(x^i) := \int_{\varkappa\Gamma} \mathrm{d}x\, \mathrm{e}^{-V(x)} x^i \tag{2.1b}$$

where the integration over $\varkappa\Gamma$ represents a suitable linear combination of integrals over contours to be explained in detail below. The function $V(x)$ is called the *potential* and will be taken here to be a polynomial of degree $d+1$,

$$V(x) := \sum_{K=1}^{d+1} u_K \frac{x^K}{K} \tag{2.2}$$

in which the coefficients $\{u_K\}$ are viewed as deformation parameters. (The constant term is suppressed since it only contributes to the overall normalization.)

This is a generalized setting which reduces to the ordinary orthogonal polynomials when the measure is positive and the contour of integration is on the real axis. We do not assume that $V(x)$ is of even degree or that it is real; the moment functional (2.1) is in general complex-valued. In this general setting the orthogonal polynomials (if they exist) are defined as follows.

Definition 2.1. *The (monic) orthogonal polynomials $\{p_n(x)\}_{n\in\mathbb{N}}$ are defined by the requirements*

$$\mathcal{L}_{\varkappa\Gamma}(p_n(x)p_m(x)) = h_n\delta_{nm} \qquad h_n \in \mathbb{C}^\times \tag{2.3a}$$

$$p_n(x) = x^n + \cdots. \tag{2.3b}$$

It is easily seen [31] that such polynomials exist if and only if the Hankel determinants formed from the moments do not vanish

$$\Delta_n(\varkappa\Gamma, V) := \det(\mathcal{L}_\Gamma(x^{i+j}))_{0\leqslant i,j\leqslant n-1} \neq 0 \qquad \forall n \in \mathbb{N}. \tag{2.4}$$

(If they do exist, equations (2.3a) and (2.3b) determine them uniquely.)

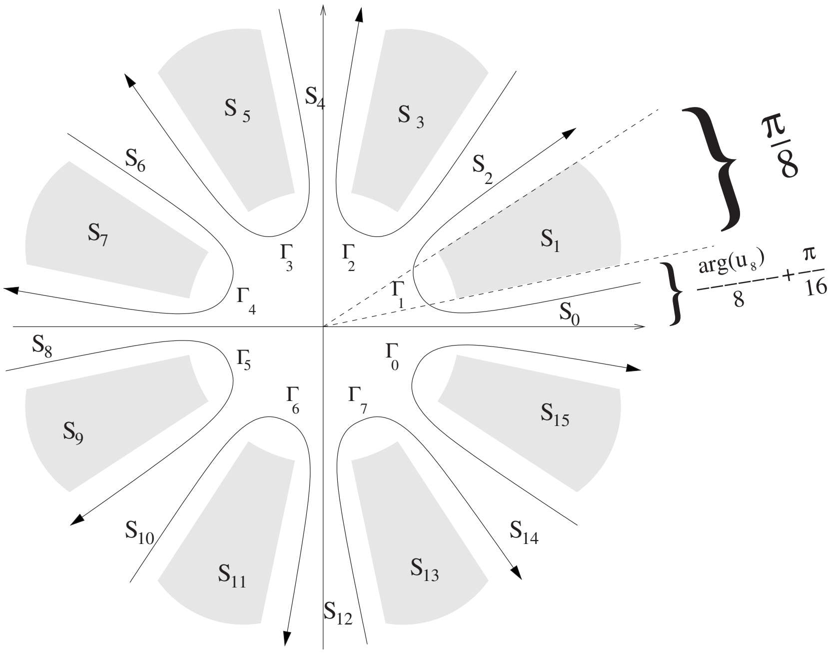

Figure 1. An example of the contours Γ_k in the case $d = 7$.

We now return to the definition of the cycle $\varkappa\Gamma$. The convergence of the integral defining the moment functional $\mathcal{L}$ implies that the contour of integration must extend to infinity in such a way that the real part of $V(x)$ diverges to $+\infty$. For the purpose of the description of $\varkappa\Gamma$ let us introduce the sectors

$$\begin{aligned} \mathcal{S}_k &:= \left\{x \in \mathbb{C},\ \arg(x) \in \left(\vartheta + \frac{(2k-1)\pi}{2(d+1)}, \vartheta + \frac{(2k+1)\pi}{2(d+1)}\right)\right\} \\ \vartheta &:= \frac{\arg(u_{d+1})}{d+1} \qquad k = 0, \ldots, 2d+1. \end{aligned} \tag{2.5}$$

These sectors are defined in such a way that $\Re(V(x)) \to +\infty$ in the even numbered sectors. Therefore, a contour should come from infinity in an even sector and return to infinity in a different even sector. Such functionals have been studied in a similar setting in [23, 24].

We define the contours Γ_k, $k \in \mathbf{Z}_d$ as the ones coming from infinity in the sector $\mathcal{S}_{2k-2}$ and returning to infinity in the sector $\mathcal{S}_{2k}$ (see figure 1). It follows from the Cauchy theorem that the corresponding $d+1$ functionals $\mathcal{L}_{\Gamma_k}$ satisfy

$$\sum_{k=0}^{d} \mathcal{L}_{\Gamma_k} = 0 \tag{2.6}$$

but it can be shown that any d of them are linearly independent [23]. In general, we may consider any linear combination of such moment functionals corresponding to a 'one-cycle' and the corresponding deformation theory under changes of complex generalized measure. The quantities $\varkappa\Gamma$ and $\mathcal{L}_{\varkappa\Gamma}$ are understood to mean

$$\varkappa\Gamma := \sum_{k=1}^{d} \varkappa_k \Gamma_k \tag{2.7a}$$

$$\mathcal{L}_{\varkappa\Gamma}(\cdot) := \sum_{k=1}^{d} \varkappa_k \mathcal{L}_{\Gamma_k}(\cdot) := \sum_{k=1}^{d} \varkappa_k \int_{\Gamma_k} \mathrm{d}x\, \mathrm{e}^{-V(x)}(\cdot) =: \int_{\varkappa\Gamma} \mathrm{d}x\, \mathrm{e}^{-V(x)}(\cdot) \quad (2.7b)$$

where $\{\varkappa_k \in \mathbb{C}\}_{k=1,\ldots,d}$ are any fixed set of constants (not all vanishing). (A similar setting, in which the contours Γ_k play the roles of boundaries between Stokes sectors, was considered, e.g., in the earlier works [6, 14, 20, 21].)

For a fixed potential $V(x)$ the Hankel determinants $\Delta_n(\varkappa\Gamma, V)$ are homogeneous polynomials in $\varkappa$ of degree n. The nonvanishing of this denumerable set of polynomials determines a set with full (Lebesgue) measure in the $\varkappa$ space and in this sense the orthogonal polynomials exist for generic $\varkappa$. (An interesting problem, which will not be addressed here, is whether the set of $\varkappa$ for which the orthogonal polynomials exist contains an open set, thus allowing for arbitrary infinitesimal deformations.)

If the contour of integration is not the real axis this setting does not derive in general from a matrix model. Nevertheless, we introduce what we call the *partition function*, in analogy with the Hermitian matrix model case [25, 31], through the following formula:

$$\mathcal{Z}_n := \int_{\varkappa\Gamma} \mathrm{d}x_1 \cdots \int_{\varkappa\Gamma} \mathrm{d}x_n \Delta(\underline{x})^2\, \mathrm{e}^{-\sum_{j=1}^{n} V(x_j)} = n!\Delta_n(\varkappa, V) \qquad \Delta(\underline{x}) := \prod_{i<j}(x_i - x_j). \quad (2.8)$$

It is well known that $\mathcal{Z}_n$ is proportional to the product of all normalization factors for the monic orthogonal polynomials in definition 2.1

$$\mathcal{Z}_n = n! \prod_{j=0}^{n-1} h_j. \quad (2.9)$$

Instead of the p_n, it is convenient to introduce the following quasi-polynomial functions:

$$\psi_n(x) = \frac{1}{\sqrt{h_n}} p_n(x)\, \mathrm{e}^{-\frac{1}{2}V(x)} \quad (2.10)$$

which are orthonormal, in the sense that

$$\int_{\varkappa\Gamma} \mathrm{d}x\, \psi_n(x)\psi_m(x) = \delta_{nm}. \quad (2.11)$$

These (quasi-)polynomials satisfy a three-term recursion relation [31]

$$x\psi_n(x) = \sum_{m=0}^{\infty} Q_{nm}\psi_m = \gamma_{n+1}\psi_{n+1} + \beta_n\psi_n + \gamma_n\psi_{n-1} \quad (2.12)$$

where

$$\gamma_n = Q_{n,n-1} = \sqrt{\frac{h_n}{h_{n-1}}} \neq 0 \qquad \forall n \in \mathbb{N} \quad \text{and} \quad \beta_n = Q_{nn}. \quad (2.13)$$

The semi-infinite matrix Q is symmetric ($Q_{nm} = Q_{mn}$), and has its nonzero entries along a diagonal band of width 3 ($Q_{mn} = 0$ if $|m-n| > 1$); i.e., it is a symmetric Jacobi matrix:

$$Q = \begin{pmatrix} \beta_0 & \gamma_1 & 0 & \ldots & \ldots \\ \gamma_1 & \beta_1 & \gamma_2 & 0 & \\ 0 & \gamma_2 & \beta_2 & \gamma_3 & \\ \vdots & 0 & \ddots & \ddots & \ddots \end{pmatrix}. \quad (2.14)$$

These recursion relations may equivalently be expressed as a two-component vector recursion

$$\begin{bmatrix} \psi_n(x) \\ \psi_{n+1}(x) \end{bmatrix} = R_n(x) \begin{bmatrix} \psi_{n-1}(x) \\ \psi_n(x) \end{bmatrix} \tag{2.15}$$

where

$$R_n(x) := \begin{bmatrix} 0 & 1 \\ -\frac{\gamma_n}{\gamma_{n+1}} & \frac{x-\beta_n}{\gamma_{n+1}} \end{bmatrix}. \tag{2.16}$$

We can also express the derivatives of the ψ_n in the basis $\{\psi_m\}_{m=0\ldots\infty}$:

$$\psi_n'(x) = \sum_{m=0}^{\infty} P_{nm}\psi_m(x) \tag{2.17}$$

where P has non-zero entries in a band of width $2d + 1$ centred on the principal diagonal. An integration by parts using equation (2.11) shows that the semi-infinite matrix P is antisymmetric. From

$$p_n' = np_{n-1} + \mathcal{O}(x^{n-2}) \tag{2.18}$$

one obtains

$$P_{nm} + \tfrac{1}{2}(V'(Q))_{nm} = 0 \qquad \text{if} \quad m \geqslant n \tag{2.19a}$$

$$P_{n,n-1} + \frac{1}{2}(V'(Q))_{n,n-1} = \frac{n}{\gamma_n} \tag{2.19b}$$

and hence

$$P = -\tfrac{1}{2}(V'(Q)_+ - V'(Q)_-) \tag{2.20}$$

where for any semi-infinite matrix A, A_+ (resp. A_-) denotes the upper (resp. lower) triangular part of A. Equation (2.20) also implies the following equations, referred to sometimes as 'string equations' [10–13, 21]:

$$[Q, P] = \mathbf{1} \qquad 0 = V'(Q)_{nn} \qquad \frac{n}{\gamma_n} = V'(Q)_{n,n-1}. \tag{2.21}$$

2.1. Differential system

It follows from (2.20) that the sum in (2.17) contains only a finite number of terms and can be written as

$$\psi_n'(x) = -\frac{1}{2}\sum_{k=1}^{d}(V'(Q)_{n,n+k}\psi_{n+k} - V'(Q)_{n,n-k}\psi_{n-k}). \tag{2.22}$$

Using equation (2.12) recursively for any $0 < k \leqslant d$ we can write ψ_{n+k} and ψ_{n-1-k} as a unique linear combination of ψ_n and ψ_{n-1} with coefficients that are polynomials in x of degree $\leqslant k$. Equation (2.22) can thus be rewritten as a 2×2 differential system:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} \psi_{n-1}(x) \\ \psi_n(x) \end{bmatrix} = \mathcal{D}_n(x)\begin{bmatrix} \psi_{n-1}(x) \\ \psi_n(x) \end{bmatrix} \qquad n \geqslant 1 \tag{2.23}$$

where $\mathcal{D}_n(x)$ is a 2×2 matrix with polynomial coefficients of degree at most $d = \deg V'$. In fact, we can choose the sequence of $\mathcal{D}_n$ to satisfy the recursive gauge transformation relations

$$\mathcal{D}_{n+1} = R_n\mathcal{D}_n R_n^{-1} + \partial_x R_n R_n^{-1}. \tag{2.24}$$

(See also [5] for similar statements for the case of two-matrix models.) This relation determines the sequence of $\mathcal{D}_n$ uniquely [4, 8, 14, 19–21]. It follows in particular that these transformations preserve the (generalized) monodromy of the linear systems (2.23) for all n.

Theorem 2.1. *The matrix $\mathcal{D}_n(x)$ is given by the formula*

$$\mathcal{D}_n(x) = \frac{1}{2}V'(x)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + \begin{bmatrix}\left(\frac{V'(Q)-V'(x)}{Q-x}\right)_{n-1,n-1} & \left(\frac{V'(Q)-V'(x)}{Q-x}\right)_{n-1,n}\\ \left(\frac{V'(Q)-V'(x)}{Q-x}\right)_{n,n-1} & \left(\frac{V'(Q)-V'(x)}{Q-x}\right)_{nn}\end{bmatrix}\begin{bmatrix}0 & -\gamma_n\\ \gamma_n & 0\end{bmatrix} \tag{2.25a}$$

$$= \frac{1}{2}V'(x)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + \gamma_n\begin{bmatrix}\left(\frac{V'(Q)-V'(x)}{Q-x}\right)_{n-1,n} & -\left(\frac{V'(Q)-V'(x)}{Q-x}\right)_{n-1,n-1}\\ \left(\frac{V'(Q)-V'(x)}{Q-x}\right)_{nn} & -\left(\frac{V'(Q)-V'(x)}{Q-x}\right)_{n,n-1}\end{bmatrix}. \tag{2.25b}$$

Proofs of this result may be found in [4, 8, 19] and earlier in [21] for the case of even potentials. In appendix A, we give an alternative proof which is based upon the deformation formulae obtained in theorem 2.2.

Remark 2.1. The matrix $\gamma_n\left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]$ appearing in the first line of equation (2.25*b*) arises naturally in the Christoffel–Darboux formula written here for the quasi-polynomials

$$(x-x')\sum_{j=0}^{n-1}\psi_j(x)\psi_j(x') = \gamma_n(\psi_n(x)\psi_{n-1}(x') - \psi_{n-1}(x)\psi_n(x'))$$

$$= \gamma_n[\psi_{n-1}(x), \psi_n(x)]\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\psi_{n-1}(x')\\ \psi_n(x')\end{bmatrix}. \tag{2.26}$$

Equation (2.24) is the starting point that allows us to explicitly compute the spectral curve of the matrix $\mathcal{D}_n(x)$ in terms of the deformation parameters $\{u_K\}_{K=1\ldots d}$ and the partition function $\mathcal{Z}_n$.

In addition to equation (2.23) we may consider the effect of deformations of the $\{u_K\}$ parameters on the orthogonal polynomials. It is well known that these give rise to isospectral deformations of the Jacobi matrix Q (Toda hierarchy equations [15, 27])

$$\frac{\partial}{\partial u_K}Q = [U_K, Q] \tag{2.27a}$$

$$\frac{\partial}{\partial u_K}\psi_n(x) = \sum_m (U_K)_{nm}\psi_m(x) \tag{2.27b}$$

$$U_K := -\frac{1}{K}\left(Q^K{}_+ - Q^K{}_-\right). \tag{2.27c}$$

Combining the recursion relations (2.12) with equations (2.27*a*)–(2.27*c*) one obtains deformation equations for the successive pairs of orthogonal quasi-polynomials

$$\frac{\partial}{\partial u_K}\begin{bmatrix}\psi_n(x)\\ \psi_{n+1}(x)\end{bmatrix} = \mathcal{U}_{K,n}(x)\begin{bmatrix}\psi_n(x)\\ \psi_{n+1}(x)\end{bmatrix}. \tag{2.28}$$

The matrices $\mathcal{U}_{K,n}(x)$ can be explicitly computed and chosen to have polynomial entries of degree $\leqslant K$ as given in the following theorem, whose proof is given in the appendix. (Similar formulae for the case of even potentials were given in [21]. The main simplification in that case is that the coefficients β_n vanish for all n.)

Theorem 2.2. *The expression for* $\mathcal{U}_{K,n}(x)$ *may be taken as*

$$\mathcal{U}_{K,n}(x) = \frac{1}{2K}\begin{bmatrix} x^K - Q^K_{n-1,n-1} & 0 \\ 0 & Q^K_{nn} - x^K \end{bmatrix} + \frac{1}{K}\gamma_n \begin{bmatrix} \left(\frac{x^K - Q^K}{x-Q}\right)_{n,n-1} & -\left(\frac{x^K - Q^K}{x-Q}\right)_{n-1,n-1} \\ \left(\frac{x^K - Q^K}{x-Q}\right)_{nn} & -\left(\frac{x^K - Q^K}{x-Q}\right)_{n,n-1} \end{bmatrix}. \tag{2.29}$$

Remark 2.2. It follows as a corollary to theorem 2.2 that

$$\operatorname{Tr}\mathcal{U}_{K,n}(x) = \frac{\partial}{\partial u_K}\ln\gamma_n. \tag{2.30}$$

It is important to note that equations (2.15), (2.23) and (2.28) do not just hold for the sequence $\{\psi_n\}$ of orthogonal quasi-polynomials but define an overdetermined system of differential-difference-deformation equations that are actually compatible in the Frobenius sense. Hence there exists, for all n, a basis of simultaneous solutions of these equations. This system of solutions can be written explicitly by adding to the ψ_n a solution of the second type as follows:

$$\Psi_n(x) = \begin{bmatrix} \psi_{n-1}(x) & \widetilde{\psi}_{n-1}(x) \\ \psi_n(x) & \widetilde{\psi}_n(x) \end{bmatrix} \tag{2.31a}$$

$$\widetilde{\psi}_n(x) = \mathrm{e}^{\frac{1}{2}V(x)}\int_{\varkappa\Gamma} \mathrm{d}x \frac{\mathrm{e}^{-\frac{1}{2}V(z)}\psi_n(z)}{x-z}. \tag{2.31b}$$

It is an easy exercise to show that these $\widetilde{\psi}_n$ satisfy the same recursion relations as the ψ_n for $n > d$ and hence satisfy the same ODE (2.23). It is also straightforward to verify that they satisfy the deformation equations (2.28). The fact that these equations are compatible implies that the generalized monodromy data (i.e. the Stokes matrices) for the operator $\partial_x - \mathcal{D}_n(x)$ are independent of the deformation parameters $\{u_K\}$ and the integer n.

We now turn to the main object of this paper: to relate the partition function (2.8) to the isomonodromic tau function (whose definition is given below). The first step is to connect the spectral curve of the matrix $\mathcal{D}_n(x)$ with the partition function.

Theorem 2.3. *The spectral curve is given by*

$$0 = \det(y\mathbf{1} - \mathcal{D}_n(x)) = y^2 - \frac{1}{2}\operatorname{Tr}\left(\mathcal{D}_n{}^2(x)\right) = y^2 - \frac{1}{4}(V'(x))^2 + \sum_{j=0}^{n-1}\left(\frac{V'(Q) - V'(x)}{Q-x}\right)_{jj} \tag{2.32a}$$

$$= y^2 - \frac{1}{4}(V'(x))^2 + n\sum_{J=1}^{d} u_{J+1}x^{J-1} + \sum_{K=0}^{d-2} x^K \sum_{J=1}^{d-K-1} u_{J+K+2} J \frac{\partial}{\partial u_J}\ln(\mathcal{Z}_n). \tag{2.32b}$$

Before proving this result, we note that the RHS contains matrix elements of Q with all indices down to $n = 0$ while, as it stands, formula (2.25b) for $\mathcal{D}_n(x)$ contains only matrix elements of Q with indices 'around' n, i.e., at distances less than $d + 1$ from Q_{nn}.

Proof of theorem 2.3. First, note that $\mathcal{D}_n(x)$ is traceless (by formula (2.25*b*)). Using equation (2.24) and the cyclicity of traces of products one finds that

$$\mathrm{Tr}(\mathcal{D}_{n+1}(x)^2) = \mathrm{Tr}(\mathcal{D}_n(x)^2) + 2\mathrm{Tr}\left(\mathcal{D}_n(x){R_n}^{-1}(x)R_n'(x)\right) + \mathrm{Tr}\left(\left(R_n'(x){R_n}^{-1}(x)\right)^2\right). \tag{2.33}$$

Now note that

$$R_n'(x){R_n}^{-1}(x) = \begin{bmatrix} 0 & 0 \\ \frac{1}{\gamma_n} & 0 \end{bmatrix} \qquad R_n^{-1}(x)R_n'(x) = \begin{bmatrix} 0 & -\frac{1}{\gamma_n} \\ 0 & 0 \end{bmatrix} \tag{2.34}$$

so the third trace term vanishes and the second one gives

$$\mathrm{Tr}(\mathcal{D}_{n+1}(x)^2) = \mathrm{Tr}(\mathcal{D}_n(x)^2) - 2\left(\frac{V'(Q) - V'(x)}{Q - x}\right)_{nn}. \tag{2.35}$$

Therefore

$$\mathrm{Tr}(\mathcal{D}_n(x)^2) = \mathrm{Tr}(\mathcal{D}_1(x)^2) - 2\sum_{j=1}^{n-1}\left(\frac{V'(Q) - V'(x)}{Q - x}\right)_{jj}. \tag{2.36}$$

Note that $\mathcal{D}_0$ is not defined. We therefore need to compute the trace of the square of the first matrix $\mathcal{D}_1$. To this end we introduce the notation

$$W(x) := \frac{V'(Q) - V'(x)}{Q - x}. \tag{2.37}$$

Now use the string equation (2.21) to compute

$$-V'(x) = [V'(Q) - V'(x)]_{00} = [W(x)(Q - x)]_{00} = (\beta_0 - x)W_{00}(x) + \gamma_1 W_{10}(x) \tag{2.38a}$$

$$\frac{1}{\gamma_1} = [V'(Q) - V'(x)]_{10} = [W(x)(Q - x)]_{10} = \gamma_1 W_{11}(x) + (\beta_0 - x)W_{10}(x). \tag{2.38b}$$

Multiplying equation (2.38*a*) by $\gamma_1 W_{10}$, equation (2.38*b*) by $\gamma_1 W_{00}$ and subtracting, we obtain (since the diagonal elements of $V'(Q)$ vanish by equation (2.21))

$$W_{00}(x) = {\gamma_1}^2 W_{11}(x)W_{00}(x) - \gamma_1 W_{10}(x)(V'(x) + \gamma_1 W_{10}(x)). \tag{2.39}$$

Computing the trace of the square of $\mathcal{D}_1(x)$, and using equation (2.39) we obtain

$$\mathrm{Tr}(\mathcal{D}_1(x)^2) = \tfrac{1}{2}V'(x)^2 - 2W_{00}(x). \tag{2.40}$$

Therefore, the full formula for the trace of the square is

$$\mathrm{Tr}(\mathcal{D}_n(x)^2) = \frac{1}{2}V'(x)^2 - 2\sum_{j=0}^{n-1}\left(\frac{V'(Q) - V'(x)}{Q - x}\right)_{jj}. \tag{2.41}$$

Remark 2.3. Note that, had we interpreted formula (2.25*b*) for $n = 0$ to mean that $\gamma_0 = 0$, we would have obtained exactly the same result.

We now expand the polynomial

$$\frac{V'(Q) - V'(x)}{Q - x} = \sum_{J=1}^{d} u_{J+1}x^{J-1}\mathbf{1} + \sum_{K=0}^{d-2} x^K \sum_{J=1}^{d-K-1} u_{J+K+2}Q^J \tag{2.42}$$

and use the fact (proved in appendix A, equation (A.5)) that

$$\sum_{j=0}^{n-1}(Q^K)_{jj} = K\,\partial_{u_K}\ln(\mathcal{Z}_n) \tag{2.43}$$

to complete the proof of the theorem. □

Theorem 2.3 relates the spectral curve of the matrix $\mathcal{D}_n(x)$ to the partition function $\mathcal{Z}_n$. The latter is known to be a τ-function for the KP hierarchy. (A short proof of this fact is given in appendix B together with an interpretation of $\mathcal{Z}_n$ as a Fredholm determinant.) On the other hand, a linear system of the form (2.23) with compatible deformation equations (2.15), (2.28) allows one to define an *isomonodromic tau function* [22]. We recall here briefly, in a way adapted to the case at hand, how to define such a tau-function.

The ODE

$$\frac{\mathrm{d}}{\mathrm{d}x}\Psi(x) = \mathcal{D}_n(x)\Psi(x) \tag{2.44}$$

has just one irregular singularity at $x = \infty$ and hence has Stokes sectors, which are just those defined in equation (2.5). Since the leading term of the singularity in $\mathcal{D}_n(x)$ is semisimple with distinct eigenvalues

$$\mathcal{D}_n(x) \simeq \frac{u_{d+1}}{2}x^d\sigma_3 + \mathcal{O}(x^{d-1}) \tag{2.45}$$

we can write in each Stokes sector $\mathcal{S}_k$ a formal asymptotic solution of the form

$$\Psi_n(x) \simeq C_n Y_n(x)\,\mathrm{e}^{T_n(x)} \tag{2.46a}$$

$$Y_n(x) := \mathbf{1} + \sum_{j=1}^{\infty} Y_{j,n}x^{-j} \tag{2.46b}$$

$$T_n(x) = \sum_{K=1}^{d+1}\frac{1}{K}T_K x^K + T_{-1}\ln(x) \tag{2.46c}$$

where the T_K are all diagonal and (in our case) traceless. In the present case, we easily find $T_n(x)$ from the tracelessness and the exponential part of the asymptotic form of the fundamental system defined in equation (2.31*b*) (cf [6, 7] for similar formulae in the case of quartic even potentials). That is,

$$T_K = \frac{1}{2}\,\mathrm{diag}(u_K, -u_K) = \frac{u_K}{2}\sigma_3 \qquad K > 0 \quad T_{-1} = n\sigma_3 \tag{2.47a}$$

$$T_n(x) = -\left(\tfrac{1}{2}V(x) - n\ln(x)\right)\sigma_3. \tag{2.47b}$$

The Stokes matrices relate the basis of solutions to these asymptotic forms in contiguous sectors. The isomonodromic tau function is then defined by the deformation equation. (See the introduction to [22] for the general context and motivations behind the theory of isomonodromic tau functions.)

$$\begin{aligned}\mathrm{d}\ln(\tau_{IM,n}) &:= \operatorname*{res}_{x=\infty}\,\mathrm{Tr}\left(Y_n^{-1}(x)Y_n'(x)\,\mathrm{d}T(x)\right)\\ &= \sum_{K=1}^{d+1}\frac{1}{2K}\operatorname*{res}_{x=\infty}\,\mathrm{Tr}\left(Y_n^{-1}(x)Y_n'(x)\sigma_3 x^{K-1}\right)\mathrm{d}u_K\end{aligned} \tag{2.48}$$

where

$$\mathrm{d} := \sum_{K=1}^{d+1}\mathrm{d}u_K\frac{\partial}{\partial u_K}. \tag{2.49}$$

We now state and prove the main theorem relating $\tau_{IM,n}(\varkappa, V)$ and $\mathcal{Z}_n(\varkappa, V)$.

Theorem 2.4. *The partition function $\mathcal{Z}_n(V)$ is related to the isomonodromic tau function by the formula*

$$\tau_{IM,n}(\varkappa, V) = \mathcal{Z}_n(\varkappa, V) u_{d+1}{}^{\frac{n^2}{d+1}} \tag{2.50}$$

up to a multiplicative factor that depends only on $\varkappa$.

Proof. We will prove that

$$\mathrm{d}\ln\mathcal{Z}_n = \mathrm{d}\ln\tau_{IM,n} - \frac{n^2}{(d+1)u_{d+1}}\,\mathrm{d}u_{d+1}.$$

We start by observing that (suppressing the dependence on x and n for brevity and denoting the derivative with respect to x by a prime)

$$\mathcal{D}_n = \Psi'\Psi^{-1} = CY'Y^{-1}C^{-1} + CYT'Y^{-1}C^{-1} \tag{2.51a}$$

$$\mathrm{Tr}\left(\mathcal{D}_n{}^2\right) = \mathrm{Tr}((Y'Y^{-1})^2) + \mathrm{Tr}(T'^2) + 2\,\mathrm{Tr}(Y^{-1}Y'T'). \tag{2.51b}$$

Consider the coefficient of x^K in $\mathrm{Tr}\big(\mathcal{D}_n{}^2\big)$, which is

$$\begin{aligned}
&\frac{1}{2}\sum_{j=0}^{K} u_{j+1}u_{k+1-j} - 2nu_{K+2} - 2\sum_{J=1}^{d-K-1} u_{J+K+2}J\frac{\partial}{\partial u_J}\ln(\mathcal{Z}_n) \overset{\text{theorem 2.3}}{=} -\operatorname*{res}_{x=\infty} x^{-K-1}\,\mathrm{Tr}\left(\mathcal{D}_n{}^2\right)\\
&\quad = \overbrace{-\operatorname*{res}_{x=\infty} x^{-K-1}\,\mathrm{Tr}((Y'Y^{-1})^2)}^{=0 \text{ because } Y'Y^{-1}=\mathcal{O}(x^{-2})} - 2\operatorname*{res}_{x=\infty} x^{-K-1}\left(\frac{1}{2}V'(x) - \frac{n}{x}\right)^2\\
&\qquad - 2\operatorname*{res}_{x=\infty} x^{-K-1}\,\mathrm{Tr}(Y^{-1}Y'T') = \frac{1}{2}\sum_{J=0}^{K} u_{J+1}u_{K+1-J} - 2nu_{K+2}\\
&\qquad - \sum_{J=1}^{d+1}\operatorname*{res}_{x=\infty} u_J\,\mathrm{Tr}(Y^{-1}Y'\sigma_3 x^{J-K-2}) = \frac{1}{2}\sum_{J=0}^{K} u_{J+1}u_{K+1-J} - 2nu_{K+2}\\
&\qquad - 2\sum_{J=1}^{d-K-1} u_{J+K+2}J\frac{\partial}{\partial u_J}\ln(\tau_{IM,n}).
\end{aligned} \tag{2.52}$$

We therefore have the relations

$$\sum_{J=1}^{d-K-1} u_{J+K+2}J\frac{\partial}{\partial u_J}\ln(\mathcal{Z}_n) \equiv \sum_{J=1}^{d-K-1} u_{J+K+2}J\frac{\partial}{\partial u_J}\ln(\tau_{IM,n}) \qquad K = 0, \ldots, d-2. \tag{2.53}$$

The vector fields appearing in equation (2.53) are just representations of the negative Virasoro generators on the space of functions of the deformation parameters $\{u_K\}$,

$$\mathbb{V}_{-K} = \sum_{J=1}^{d+1-K} u_{J+K}J\frac{\partial}{\partial u_J}. \tag{2.54}$$

Equation (2.53) can therefore be rewritten as

$$\mathbb{V}_{-K}\ln(\mathcal{Z}_n(V)) = \mathbb{V}_{-K}\ln(\tau_{IM,n}(V)) \qquad 2 \leqslant K \leqslant \cdots \leqslant d. \tag{2.55}$$

Since the vector fields $\mathbb{V}_0, \ldots, \mathbb{V}_{-d}$ are linearly independent

$$\mathbb{V}_0 \wedge \ldots \wedge \mathbb{V}_{-d} = (d+1)!(u_{d+1})^{d+1}\partial_{u_1} \wedge \ldots \wedge \partial_{u_{d+1}} \tag{2.56}$$

and $\mathcal{Z}_n(V)$, $\tau_{IM,n}$ depend on the $d+1$ parameters $u_1, \ldots, u_{d+1}$, we may relate them up to a multiplicative constant if we determine the effect of the two remaining operators. Now $\mathbb{V}_0$ and $\mathbb{V}_{-1}$ generate dilations and translations respectively

$$\mathbb{V}_0 V(x) = x\partial_x V(x) \qquad \mathbb{V}_{-1} V(x) = \partial_x V(x). \tag{2.57}$$

Moreover, it is clear from its definition (equation (2.8)) that the partition function must satisfy

$$\mathbb{V}_{-1}\mathcal{Z}_n = 0 \qquad \mathbb{V}_0\mathcal{Z}_n = -n^2\mathcal{Z}_n. \tag{2.58}$$

On the other hand, the definition (2.48) of the isomonodromic tau function implies that it is invariant under translations and dilations of x so that

$$\mathbb{V}_0\tau_{IM,n} = \mathbb{V}_{-1}\tau_{IM,n} = 0. \tag{2.59}$$

It follows that

$$\ln(\mathcal{Z}_n) = \ln(\tau_{IM,n}) - \frac{n^2}{d+1}\ln(u_{d+1}) + \text{const} \tag{2.60}$$

which gives the proof of the theorem up to a constant multiplicative factor. □

Remark 2.4. Theorem 2.4 was proved in the special case in which $V(x) = g_1x^2 + g_2x^4$ in [20].

Remark 2.5. For any fixed n, the set of $\varkappa$ for which orthogonal polynomials exist up to degree n is an open set because it is the complement of the vanishing locus of n homogeneous polynomials

$$\Delta_m(\varkappa, V) \neq 0 \qquad \forall m \leqslant n. \tag{2.61}$$

Thus we could also study the deformations with respect to parameters $\varkappa$. However, these are not isomonodromic deformations because they change the data of the Riemann–Hilbert problem that are naturally associated with equation (2.44). A similar approach in the case of even potentials can be found in [21].

Acknowledgments

The authors would like to thank A Its and A Orlov for helpful remarks and for drawing attention to related earlier works. This work is supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC) and the Fonds FCAR du Québec.

Appendix A

In this appendix we give proofs of theorems 2.1 and 2.2. First, we consider the deformation equations, which determine the variation of the orthogonal polynomials when the potential $V(x)$ is varied.

Let us first expand the variation of $\psi_n(x)$ with respect to u_K in the basis $\{\psi_n\}$:

$$\frac{\partial}{\partial u_K}\psi_n(x) = \sum_{m=0}^{\infty} (U_K)_{nm}\, \psi_m(x). \tag{A.1}$$

The semi-infinite matrix U_K is antisymmetric (as seen from differentiating equation (2.11)):

$$(U_K)_{nm} = -(U_K)_{mn}. \tag{A.2}$$

Taking the derivative of equation (2.10) with respect to u_K and using $\partial p_n/\partial u_K = O(x^{n-1})$ and equation (A.2) we obtain

$$U_K = -\frac{1}{2K}\left((Q^K)_+ - (Q^K)_-\right). \tag{A.3}$$

In particular, the semi-infinite matrix U_K is of finite band size:

$$(U_K)_{nm} = 0 \qquad \text{if} \quad |n-m| > K \tag{A.4}$$

and therefore equation (A.1) contains only a finite sum. The diagonal part yields (from equation (2.10))

$$\frac{1}{K}(Q^K)_{nn} = \frac{\partial \ln h_n}{\partial u_K} \quad \Rightarrow \quad \frac{\partial}{\partial u_K}\ln \mathcal{Z}_n = \frac{1}{K}\sum_{i=0}^{n-1}(Q^K)_{ii}. \tag{A.5}$$

The semi-infinite matrix U_K can be used to form a differential system in the same way as for the ∂_x equations, by expressing any ψ_m in terms of ψ_n and ψ_{n-1} by means of (2.12). We then obtain an equation

$$\frac{\partial}{\partial u_K}\begin{pmatrix}\psi_{n-1}(x)\\ \psi_n(x)\end{pmatrix} = \mathcal{U}_{K,n}(x)\begin{pmatrix}\psi_{n-1}(x)\\ \psi_n(x)\end{pmatrix} \tag{A.6}$$

where $\mathcal{U}_{K,n}(x)$ is a 2×2 matrix with polynomial coefficients of degree $\leqslant K$. In particular, it is easy to compute $\mathcal{U}_{1,n}(x)$:

$$\mathcal{U}_{1,n}(x) = -\frac{1}{2}\begin{pmatrix}\beta_{n-1}-x & 2\gamma_n\\ -2\gamma_n & x-\beta_n\end{pmatrix}. \tag{A.7}$$

The general form of the matrices $\mathcal{U}_{K,n}$ is given in theorem 2.2 which we now prove.

Proof of theorem 2.2. For any semi-infinite matrix A, let A_l denote the lth diagonal above (or below, if $l < 0$) the main diagonal. Recall that A_+ (resp. A_-) denotes the strictly upper (resp. strictly lower) triangular part of A.

Since the matrix Q has only three non-vanishing diagonals, we have

$$\begin{aligned}(Q^{K+1})_+ &= (Q^KQ)_+ = (Q^KQ_1)_+ + (Q^KQ_0)_+ + (Q^KQ_{-1})_+\\ &= \left((Q^K)_+ + (Q^K)_0\right)Q_1 + (Q^K)_+Q_0 + \left((Q^K)_+ - (Q^K)_1\right)Q_{-1}\\ &= (Q^K)_+Q + (Q^K)_0Q_1 - (Q^K)_1Q_{-1}.\end{aligned} \tag{A.8}$$

This, combined with a similar calculation for $(Q^{K+1})_-$, implies

$$Q_+^{K+1} - Q_-^{K+1} = \left(Q_+^K - Q_-^K\right)Q + \left(Q_0^K + Q_{-1}^K\right)Q_1 - \left(Q_1^K + Q_0^K\right)Q_{-1} \tag{A.9}$$

which, componentwise, reads (using equation (A.3)):

$$\begin{aligned}-2(K+1)\frac{\partial}{\partial u_{K+1}}\psi_n(x) &= -2K\frac{\partial}{\partial u_K}x\psi_n(x) + (Q^K)_{nn}(\gamma_{n+1}\psi_{n+1} - \gamma_n\psi_{n-1})\\ &\quad + (\gamma_n(Q^K)_{n,n-1} - \gamma_{n+1}(Q^K)_{n,n+1})\psi_n.\end{aligned} \tag{A.10}$$

Using equation (2.12)

$$\begin{aligned}-2(K+1)\frac{\partial}{\partial u_{K+1}}\psi_n(x) &= -2K\frac{\partial}{\partial u_K}x\psi_n(x) + (Q^K)_{nn}((x-\beta_n)\psi_n - 2\gamma_n\psi_{n-1})\\ &\quad + \left(\gamma_n(Q^K)_{n,n-1} - \gamma_{n+1}(Q^K)_{n,n+1}\right)\psi_n\end{aligned} \tag{A.11a}$$

$$\begin{aligned}-2(K+1)\frac{\partial}{\partial u_{K+1}}\psi_{n-1}(x) &= -2K\frac{\partial}{\partial u_K}x\psi_{n-1}(x) + (Q^K)_{n-1,n-1}(2\gamma_n\psi_n\\ &\quad - (x-\beta_{n-1})\psi_{n-1}) + \left(\gamma_{n-1}(Q^K)_{n-1,n-2} - \gamma_n(Q^K)_{n,n-1}\right)\psi_{n-1}\end{aligned} \tag{A.11b}$$

which can be further simplified using

$$Q_{nn}^{K+1} = \gamma_{n+1} Q_{n,n+1}^{K} + \beta_n Q_{nn}^{K} + \gamma_n Q_{n,n-1}^{K} \tag{A.12a}$$

$$Q_{n-1,n-1}^{K+1} = \gamma_{n-1} Q_{n-1,n-2}^{K} + \beta_{n-1} Q_{n-1,n-1}^{K} + \gamma_n Q_{n,n-1}^{K}. \tag{A.12b}$$

We thus get a recursion formula for $\mathcal{U}_{K,n}(x)$:

$$-2(K+1)\mathcal{U}_{K+1,n}(x) + 2Kx\mathcal{U}_{K,n}(x) = \begin{pmatrix} -(xQ^K - Q^{K+1})_{n-1,n-1} & 0 \\ 0 & (xQ^K - Q^{K+1})_{nn} \end{pmatrix} + 2\gamma_n \begin{pmatrix} -Q_{n-1,n}^K & Q_{n-1,n-1}^K \\ -Q_{nn}^K & Q_{n,n-1}^K \end{pmatrix} \tag{A.13}$$

from which the result follows by induction on K and using equation (A.7). □

As a corollary to theorem 2.2 we can complete the proof of theorem 2.1.

Proof of theorem 2.1. Since $V(x)$ is a polynomial of degree $d+1$, equation (2.20) implies that

$$P = \sum_{K=0}^{d} u_{K+1} K U_K \tag{A.14}$$

and therefore

$$\mathcal{D}_n(x) = \sum_{K=1}^{d} u_{K+1} K \mathcal{U}_{K,n}(x). \tag{A.15}$$

Using theorem 2.2 and equation (2.21), one immediately finds the expression (2.25b) for $\mathcal{D}_n(x)$. □

Appendix B

We give here a simple proof of the fact that the partition function $\mathcal{Z}_n(V)$ is a KP tau function in the sense of [29, 30]; that is, the determinant of a projection operator acting on elements of a suitably defined Grassmannian, evaluated along the orbits of an Abelian group. The interest in rederiving this well-known result is that the method used, which just involves a simple formal matrix calculation, naturally leads to a new interpretation of the partition function as the Fredholm determinant of a simple integral operator.

Let

$$\mathcal{H} = \mathcal{H}_n + \mathcal{H}_+ \tag{B.1}$$

where $\mathcal{H}_n = \mathrm{span}\{z^j\}_{0\leqslant j\leqslant n-1}$ and $\mathcal{H}_+$ is a suitably defined completion (whose details are irrelevant for this sort of formal computation) of the space $\mathrm{span}\{z^j\}_{j\geqslant n}$, with $z \in \mathbb{C}$ viewed as a variable defined along our cycle $\varkappa\Gamma$. (Note that we could have included the negative powers $\{z^j\}_{j<0}$ in the definition of $\mathcal{H}$, but for the case at hand, these contribute nothing to the computation of the tau-function and it is simplest to suppress them from the start.) Denote by $\widetilde{w}_0$ the $Gl(\mathcal{H})$ element defined by

$$\begin{aligned} \widetilde{w}_0 :\quad & \mathcal{H} \longrightarrow \mathcal{H} \\ \widetilde{w}_0 :\quad & \sum_{j\geqslant 0} a_j z^j \mapsto \sum_{j\geqslant 0} a_j p_j(z) \end{aligned} \tag{B.2}$$

where $\{p_j(z)\}$ are the monic orthogonal polynomials associated with the moment functional $\mathcal{L}_{\varkappa\Gamma}$ for an initial value of the parameters $\{u_k\}$. Within the natural basis $\{z^j\}_{j\in\mathbb{N}}$ the group element $\widetilde{w}_0$ is represented by the upper triangular matrix

$$\widetilde{W}_0 := \begin{array}{c} \begin{array}{cc} n & + \end{array} \\ \left[\begin{array}{c|c} W_n & W_{n+} \\ \hline 0 & W_+ \end{array}\right] \end{array} \begin{array}{c} n \\ \hline + \end{array} \tag{B.3}$$

whose jth column consists of the expansion coefficients in

$$p_j(z) := \sum_{k=0}^{j} W_{kj} z^k. \tag{B.4}$$

(Note that the blocks W_n, W_+ are upper triangular with diagonal elements all equal to 1.)

We consider the Grassmannian $Gr_+(\mathcal{H})$ of subspaces of $\mathcal{H}$ that are 'commensurable' with $\mathcal{H}_+ \subset \mathcal{H}$ in the sense of [30], and the commuting flows induced on it by the action of the Abelian group

$$\Gamma^+ := \left\{ \widetilde{\gamma}(\mathbf{t}) = \exp\left(\sum_{J\geqslant 1} \frac{1}{J} t_J z^J\right)\right\} \subset Gl(\mathcal{H}) \tag{B.5}$$

(where $\mathbf{t} = (t_1, t_2, \ldots)$), acting by multiplication:

$$\begin{array}{ccc} \Gamma^+ \times \mathcal{H} & \longrightarrow & \mathcal{H} \\ (\widetilde{\gamma}(\mathbf{t}), f) & \mapsto & \widetilde{\gamma}(\mathbf{t}) f. \end{array} \tag{B.6}$$

Within the standard basis, the flows on $\mathcal{H}$ are represented by semi-infinite lower triangular matrices

$$\gamma(\mathbf{t}) := \left[\begin{array}{c|c} A & 0 \\ \hline B & C \end{array}, \right] \tag{B.7}$$

where the diagonal blocks are again lower triangular with 1 along the diagonal. The initial element $w_0 \in Gr_+(\mathcal{H})$ is taken as

$$w_0 := \mathrm{span}\{p_j(z)\}_{j\geqslant n}. \tag{B.8}$$

In homogeneous coordinates, relative to the standard basis, this is represented by the $(n+\infty)\times\infty$ rectangular matrix

$$W_0 := \widetilde{W}_0 \begin{bmatrix} 0 \\ \mathbf{1}_+ \end{bmatrix}. \tag{B.9}$$

The induced flow on $Gr_+(\mathcal{H})$ takes w_0 to the element with homogeneous coordinates

$$W(\mathbf{t}) = \gamma(\mathbf{t}) W_0. \tag{B.10}$$

The corresponding KP tau function $\tau_{W_0}(\mathbf{t})$ is given by the Plücker coordinate corresponding to the subspace

$$\tau_{W_0}(\mathbf{t}) = \det(B W_{n+} + C W_+). \tag{B.11}$$

Since all diagonal blocks have unit determinants, this equals

$$\tau_{W_0}(\mathbf{t}) = \det\left(\mathbf{1}_\infty + B W_{n+} W_+{}^{-1} C^{-1}\right) = \det\left(\mathbf{1}_n + W_{n+} W_+{}^{-1} C^{-1} B\right) \tag{B.12}$$

where the second equality, expressing $\tau_{W_0}(\mathbf{t})$ as a finite $n \times n$ determinant, follows from the Weinstein–Aronszajn identity.

The partition function $\mathcal{Z}_n$ in (2.8) is also expressible as an $n \times n$ determinant of the matrix of moments $\mathfrak{M}$

$$\mathfrak{M}_{ij} := \mathcal{L}_{\varkappa\Gamma}(x^{i+j}) \tag{B.13}$$

which in our notation may be expressed at the parameter values

$$u_J = u_J^{(0)} + t_J \qquad J = 1, \ldots, d+1 \tag{B.14}$$

as

$$\mathfrak{M} = \mathfrak{M}_0 \gamma^{-1}(\mathbf{t}) \tag{B.15}$$

where

$$\mathbf{t} := (t_1, t_2, \ldots, t_{d+1}, 0, 0, \ldots) \tag{B.16a}$$

$$\mathfrak{M}_0 = W_0^{t\,-1} H_0 W_0{}^{-1} \tag{B.16b}$$

is the initial value of $\mathfrak{M}$ (at $\mathbf{t} = 0$) and

$$H_0 := \operatorname{diag}(h_0, h_1, \ldots) = \begin{bmatrix} H_{0,n} & 0 \\ 0 & H_{0,+} \end{bmatrix} \tag{B.17}$$

is the initial value of the matrix of normalization factors.

Let

$$\widehat{\iota}_n : \mathcal{H}_n \hookrightarrow \mathcal{H} \qquad \widehat{\pi}_n : \mathcal{H} \to \mathcal{H}_n \tag{B.18}$$

be the natural injection and projection represented by the matrices

$$\iota_n := \begin{bmatrix} \mathbf{1}_n \\ 0 \end{bmatrix} \qquad \pi_n := [\mathbf{1}_n \quad 0]. \tag{B.19}$$

Then, by equation (2.8)

$$\frac{1}{n!}\mathcal{Z}_n = \det(\pi_n \mathfrak{M} \iota_n) = \det\left(\pi_n W_0^{t\,-1} H_0 W_0{}^{-1} \gamma^{-1} \iota_n\right) \tag{B.20a}$$

$$= \det(H_{0,n}) \det\left(\mathbf{1}_n + W_{n+} W_+{}^{-1} C^{-1} B\right) \tag{B.20b}$$

where the second line follows from the fact that

$$\det(A) = \det(C) = \det(W_n) = \det(W_+) = 1. \tag{B.21}$$

Therefore

$$\tau_{W_0}(\mathbf{t}) = \frac{\mathcal{Z}_n(\mathbf{t})}{\mathcal{Z}_n(0)}. \tag{B.22}$$

□

We may now use this calculation to deduce a representation of $\mathcal{Z}_n(\mathbf{t})$ as a Fredholm determinant. First note that from the definitions of the matrices A, B, C, W_{n+}, W_+, we may view these as representations of the operators

$$\begin{aligned} \widehat{B} : \quad & \mathcal{H}_n \quad \longrightarrow \quad \mathcal{H}_+ \\ \widehat{B} : \quad & \sum_{j=0}^{n-1} a_j z^j \quad \mapsto \quad \sum_{j=0}^{n-1} a_j (\widetilde{\gamma}(\mathbf{t}) z^j)_{\geqslant n} \end{aligned} \tag{B.23a}$$

$$\begin{aligned} \widehat{C} : \quad & \mathcal{H}_+ \quad \longrightarrow \quad \mathcal{H}_+ \\ \widehat{C} : \quad & \sum_{j\geqslant n} a_j z^j \quad \mapsto \quad \sum_{j\geqslant n} a_j \widetilde{\gamma}(\mathbf{t}) z^j \end{aligned} \tag{B.23b}$$

$$\begin{aligned} \widehat{W}_+ : \quad & \mathcal{H}_+ \quad \longrightarrow \quad \mathcal{H}_+ \\ \widehat{W}_+ : \quad & \sum_{j\geqslant n} a_j z^j \quad \mapsto \quad \sum_{j\geqslant n} a_j (p_j(z))_{\geqslant n} \end{aligned} \tag{B.23c}$$

$$\begin{array}{llll} \widehat{W}_{n+}: & \mathcal{H}_+ & \longrightarrow & \mathcal{H}_n \\ \widehat{W}_{n+}: & \sum_{j\geqslant n} a_j z^j & \mapsto & \sum_{j\geqslant n} a_j (p_j(z))_{<n} \end{array} \tag{B.23d}$$

where $(f(z))_{\geqslant n}$ and $(f(z))_{<n}$ denote the parts of the Taylor series expansion of $f \in \mathcal{H}$ lying in $\mathcal{H}_+$ and $\mathcal{H}_n$, respectively. It follows that the operator

$$\widehat{K} : \mathcal{H}_+ \to \mathcal{H}_+ \tag{B.24}$$

represented above by the matrix appearing in equation (B.12):

$$-BW_{n+}{W_+}^{-1}C^{-1} \tag{B.25}$$

is just the rank-n Fredholm integral operator

$$\widehat{K} f(z) = \int_{\varkappa\Gamma} \mathrm{d}w\, \mathrm{e}^{-\sum_{k=1}^{d+1} \frac{u_K^{(0)}}{K} w^K} K(z,w) f(w) \qquad f \in \mathcal{H}_+ \tag{B.26}$$

with the kernel

$$K(z,w) = \sum_{j=0}^{n-1} \frac{1}{h_j} \left(\mathrm{e}^{\sum_{K=1}^{d+1} t_K z^K/K} p_j(z) \right)_{\geqslant n} p_j(w)\, \mathrm{e}^{-\sum_{K=1}^{d+1} t_K w^K/K} \tag{B.27}$$

and hence, by equation (B.12),

$$\tau_{W_0}(\mathbf{t}) = \det(\mathbf{1} - \widehat{K}). \tag{B.28}$$

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